Strong approximations in the dependent setting

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I. Strong approximations for the partial sums process

II. Strong approximations for the empirical process

I. On strong approximation for the partial sums

- When $(X_i)_{i\geq 1}$ is a sequence of iid centered real-valued random variables with a finite second moment, the ASIP says that a sequence $(Z_i)_{i\geq 1}$ of iid centered Gaussian variables may be constructed is such a way that

$$\sup_{1\leq k\leq n}\Big|\sum_{i=1}^{k}(X_i - Z_i)\Big| = o(b_n) \text{ almost surely,}$$

where $b_n = (n\log\log n)^{1/2}$ (Strassen (1964)).

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  where $b_n = (n\log\log n)^{1/2}$ (Strassen (1964)).

- When $(X_i)_{i\geq 1}$ is assumed to be in addition in $\mathbf{L}^p$ with $p > 2$, then we can obtain rates in the ASIP:

$$b_n = n^{1/p}$$

  (see Major (1976) for $p \in]2, 3]$ and Komlós, Major and Tusnády for $p > 3$).

The construction of Major.

- Choose an appropriate numerical sequence $0 = n_0 < n_1 < \cdots < n_k < \ldots$ and let

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- The r.v.'s $(S_{n_k} - S_{n_{k-1}}, T_{n_k} - T_{n_{k-1}})_{k \geq 1}$ are constructed in such a way that there are independent.

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- The r.v.'s $(S_{n_k} - S_{n_{k-1}}, T_{n_k} - T_{n_{k-1}})_{k\ge 1}$ are constructed in such a way that there are independent.
- *First construction*: Let $F_k(x) = \mathbf{P}(S_{n_k} - S_{n_{k-1}} \le x)$ and let $(\delta_i)_{i\ge 1}$ be a sequence of iid r.v.'s $\sim \mathcal{U}([0,1])$. Assume that $\mathbf{E}(X_1^2) = 1$. Define then
$$S_{n_k} - S_{n_{k-1}} = F_k^{-1}(\delta_k) \text{ and } T_{n_k} - T_{n_{k-1}} = (n_k - n_{k-1})^{1/2}\Phi^{-1}(\delta_k)$$

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- *Second construction*: let $(\delta_i)_{i\geq 1}$ be a sequence of iid r.v.'s $\sim \mathcal{U}([0,1])$ independent of the sequence (X_i) (enlarge the probability space if necessary). Let $\widetilde{S}_k = S_{n_k} - S_{n_{k-1}}$. Define then

$$T_{n_k} - T_{n_{k-1}} = (n_k - n_{k-1})^{1/2}\Phi^{-1}(F_k(\widetilde{S}_k - 0) + \delta_k(F_k(\widetilde{S}_k) - F_k(\widetilde{S}_k - 0)))$$

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- Setting $\widetilde{T}_k = T_{n_k} - T_{n_{k-1}}$, both constructions satisfy

$$\|\widetilde{S}_k - \widetilde{T}_k\|_2^2 = \int_0^1 (F_k^{-1}(x) - \Phi^{-1}_{n_k - n_{k-1}}(x))^2 dx$$
$$= W_2^2(P_{\widetilde{S}_k}, G_{n_k - n_{k-1}})\,.$$

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- If the random variables are in $\mathbf{L}^p$ for $p > 2$, we then get that

$$\|S_{n_k} - S_{n_{k-1}} - (T_{n_k} - T_{n_{k-1}})\|_2^2 = O(1 \vee (n_k - n_{k-1})^{2-p/2}),$$

interpolating the results by Ibragimov (66), Sakhanenko (85), Rio (09)

The construction of Major.

- Assume that the sequence has many moments (more than 4) and allow some $\log$ in the symbol $O(\cdot)$. Then by the Kolmogorov inequality, we get that almost surely
$$\sup_{j\leq k} |S_{n_j} - T_{n_j}| = O(k^{1/2})$$
and we also have (if $n_k - n_{k-1}$ is monotone and goes to infinity fast enough)
$$\sup_{j\leq k-1} \sup_{n_j<n<n_{j+1}} |S_n - S_{n_j}| = O((n_k - n_{k-1})^{1/2})$$

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- With this construction no way to get better rate than $n^{1/4}$.

The dependent setting. M. and Rio (2012).

- Let $m_k = n_k - n_{k-1}$, $\widetilde{S}_k = S_{n_k} - S_{n_{k-1}}$ and $\widetilde{T}_k = T_{n_k} - T_{n_{k-1}}$.

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$$T_{n_k} - T_{n_{k-1}} = \sigma\sqrt{m_k}\Phi^{-1}(\widetilde{F}_k(\widetilde{S}_k - 0) + \delta_k(\widetilde{F}_k(\widetilde{S}_k) - \widetilde{F}_k(\widetilde{S}_k - 0)))$$

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- We are lead to estimate

$$\|\widetilde{S}_k - \widetilde{T}_k\|_2^2 = \mathbf{E}\int_0^1 (\widetilde{F}_k^{-1}(x) - \Phi_{\sigma^2 m_k}^{-1}(x))^2 dx = \mathbf{E}\big(W_2^2(P_{\widetilde{S}_k|\mathcal{F}_{n_{k-1}}}, G_{\sigma^2 m_k})\big).$$

A bound for the ”conditional” W_2

- P and Q two probability laws on $\mathbb{R}$ with d.f. F et G.

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- *Lemma (M. and Rio (2012))* : Let Z be a r.v. with values in $(E, \mathcal{L}(E), m)$ (a purely non atomic Lebesgue space) and let $\mathcal{F} = \sigma(Z)$. Let U and V r.v.'s with V independent of $\mathcal{F}$. Let $\sigma^2 > 0$ and $N \sim \mathcal{N}(0,\sigma^2)$ independent of $\sigma(Z,U,V)$. Then

$$\mathbb{E}\big(W_2^2(P_{U|\mathcal{F}}, P_V)\big) \leq 16 \sup_{f \in \Lambda_2(E)} \mathbb{E}\big(f(U+N,Z) - f(V+N,Z)\big) + 8\sigma^2\,,$$

where $\Lambda_2(E)$ is the set of functions $f : \mathbb{R} \times E \to \mathbb{R}$ wrt $\mathcal{L}(\mathbb{R} \times E)$ and $\mathcal{B}(\mathbb{R})$, such that $f(\cdot, z) \in \Lambda_2$ et $f(0,z) = f'(0,z) = 0$ for $z \in E$.

Rates in the ASIP under α-dependence

- Let $\mathcal{F}_0 = \sigma(X_i, i \leq 0)$ and the dependence coefficients: $\alpha_2(0) = 1$,

$$\alpha_2(k) = \sup_{i \geq j \geq k} \sup_{(s,t) \in \mathbf{R}^2} \| \mathbf{P}(X_i \leq t, X_j \leq s | \mathcal{F}_0) - \mathbf{P}(X_i \leq t, X_j \leq s) \|_1 \, , k \geq 1$$

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- *Proposition (M. and Rio (2012))* : Assume that the series $\mathbf{E}(X_0^2) + 2\sum_{k \geq 1} \mathbf{E}(X_0 X_k)$ is convergent to some nonnegative real σ^2. If $\sigma^2 > 0$, then there exists a positive constant C depending on σ^2 such, that for any $n > 0$,

$$\mathbf{E}\big(W_2^2(P_{S_n|\mathcal{F}_0}, G_{n\sigma^2})\big) \leq C n^{1/2} \int_0^1 Q(u) R(u) (R(u) \wedge n^{1/2}) du \, ,$$

with $Q(u) = \inf\{t \geq 0 : \mathbf{P}(|X_0| > t) \leq u\}$, $R(u) = \alpha_2^{-1}(u)(Q(u) \vee 1)$.

Rates under α-dependence

- *Theorem (M. and Rio (2012))* : Let $p \in]2,3]$. Assume that

$$\sum_{k\geq 1} k^{p-2} \int_0^{\alpha_2(k)} Q^p_{|X_0|}(u)du < \infty\,.$$

Then the series $\mathbf{E}(X_0^2) + 2\sum_{k\geq 1}\mathbf{E}(X_0X_k)$ is convergent to some nonnegative real σ^2 and there exists a sequence $(Z_i)_{i\geq 1}$ of iid r.v.'s $\sim \mathcal{N}(0,\sigma^2)$ such that almost surely

$$\sup_{k\leq n} |S_k - T_k| = \begin{cases} 0(n^{1/p}(\log n)^{1/2-1/p}) & \text{if } p \in]2,3[\\ 0(n^{1/3}(\log n)^{1/2}(\log\log n)^{\epsilon+1/3}) & \text{if } p = 3 \text{ for any } \epsilon > 0\,. \end{cases}$$

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- For bounded r.v. the condition holds if $\alpha_2(n) = O(n^{1-p}(\log n)^{-1-\varepsilon})$ whereas as the condition in Shao and Lu (87) requires $\alpha(n) = 0(n^{-p})$.

Dynamical systems (1)

- For γ in $]0,1[$, consider the intermittent map T_γ from $[0,1]$ to $[0,1]$

$$T_\gamma(x) = \begin{cases} x(1+2^\gamma x^\gamma) & \text{if } x \in [0,1/2[\\ 2x-1 & \text{if } x \in [1/2,1]\,. \end{cases}$$

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- Let Q_γ be the Perron-Frobenius operator of T_γ with respect to ν_γ defined by

$$\nu_\gamma(f.g \circ T_\gamma) = \nu_\gamma(Q_\gamma(f).g)$$

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- Let $(Y_i)_{i\geq 0}$ be a stationary the Markov chain with invariant measure ν_γ and transition Kernel Q_γ.
- On $([0,1],\nu_\gamma)$, $(T_\gamma, T_\gamma^2,\ldots,T_\gamma^k) =^{\mathcal{L}} (Y_k,\ldots,Y_2,Y_1)$ (see Hennion and Hervé (2001)).

Dynamical systems (2)

- Any information on the law of $\sum_{i=1}^{n}(f \circ T_\gamma^i - \nu_\gamma(f))$ can be obtained by studying the law of $\sum_{i=1}^{n}(f(Y_i) - \nu_\gamma(f))$. The reverse time property cannot be directly used to transfer almost sure results!

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- For every $n > 0$, $\alpha_{2,\mathbf{Y}}(n) \leq Cn^{(\gamma-1)/\gamma}$ (Dedecker, Gouëzel and M. (2010)).
- Let $\gamma \in]1/3, 1/2[$ and f be a function of bounded variation. Then the series

$$\sigma^2(f) = \nu_\gamma((f - \nu_\gamma(f))^2) + 2\sum_{k>0} \nu_\gamma((f - \nu_\gamma(f)) f \circ T_\gamma^k)$$

converges absolutely to some nonnegative number $\sigma^2(f)$ and, for any $\varepsilon > 0$, there exists a sequence $(Z_i^*)_{i\geq 1}$ of iid random variables with law $N(0, \sigma^2(f))$ such that

$$\sup_{k\leq n} \Big|\sum_{i=1}^k (f \circ T_\gamma^i - \nu_\gamma(f) - Z_i^*)\Big| = O(n^\gamma (\log n)^{1/2} (\log\log n)^{(1+\varepsilon)\gamma}) \text{ a.s.}$$

II. Strong approximation for the empirical process

- Let $X = (X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of r.v. in $\mathbb{R}^d$ with common distribution function F. Define the empirical process of X by

$$R_X(s,t) = \sum_{1\le k\le t} \left(\mathbf{1}_{X_k\le s} - F(s)\right), \; s\in\mathbb{R}^d, \; t\in\mathbb{R}^+ .$$

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- For $d=1$ and iid r.v's X_i, Kiefer (1972) constructed a continuous centered Gaussian process K_X with

$$\mathbb{E}\big(K_X(s,t)K_X(s',t')\big) = (t\wedge t')(F(s\wedge s') - F(s)F(s'))$$

in such a way that

$$\sup_{(s,t)\in\mathbb{R}\times[0,1]} |R_X(s,[nt]) - K_X(s,[nt])| = O(a_n) \quad \text{a.s.} \qquad (*)$$

with $a_n = n^{1/3}(\log n)^{2/3}$.

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- For the empirical distribution function and Tusnády type results, Rio (96) obtained the rate $O\big(n^{5/12}(\log n)^{c(d)}\big)$ for iid random variables with the uniform distribution.
- Up to now, the best known rates for the strong approximation by a Kiefer process are of the order $n^{1/3}$ for $d = 2$.

Absolutely regular sequences: a lower bound

- Let $\mathbf{X}_k = (X_j, j \geq k)$ and

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- (i) $0 < \liminf_{n\to+\infty} n^{p-1}\beta_n \leq \limsup_{n\to+\infty} n^{p-1}\beta_n < \infty$.
- (ii) There exists a positive constant C such that, for any construction of a sequence $(G_n)_{n>0}$ of continuous Gaussian processes on $[0,1]$

$$(a) \qquad \liminf_{n\to\infty} n^{-1/p}\mathbb{E}\Big(\sup_{s\in(0,1]} |R_X(s,n) - G_n(s)|\Big) \geq C.$$

A lower bound (to continue)

- Furthermore

$$(b) \qquad \limsup_{n\to\infty} (n \log n)^{-1/p} \Big(\sup_{s\in(0,1]} |R_X(s,n) - G_n(s)| \Big) > 0 \text{ a.s.}$$

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- Π has distribution function $F(x) = x^a$. Setting $X_i = \xi_i^a$ we obtain a stationary Markov chain $(X_i)_{i\in\mathbb{Z}}$ of random variables with uniform distribution over $[0,1]$ and adequate rate of β-mixing.

Upper bound: Dedecker, M., Rio (12)

- Let $(X_i)_{i\in\mathbb{Z}}$ be a strictly stationary sequence of random variables in $\mathbb{R}^d$. Let F_j be the distribution function of the j-th marginal of X_0. Assume that $\beta_n = O(n^{1-p})$ for some $p \in]2, 3]$. Then

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- for all (s, s') in $\mathbb{R}^{2d}$, $\Lambda_X(s,s') = \sum_{k\in\mathbb{Z}} \mathrm{Cov}(\mathbf{1}_{X_0\le s}, \mathbf{1}_{X_k\le s'})$ converges absolutely.

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- For any $(s,s') \in \mathbb{R}^{2d}$ and (t,t') in $\mathbb{R}^+ \times \mathbb{R}^+$, let

$$\Gamma_X(s,s',t,t') = \min(t,t')\Lambda_X(s,s')$$

Then enlarging Ω if necessary, there exists a centered Gaussian process K_X with covariance function Γ_X, whose sample paths are almost surely uniformly continuous with respect to the pseudo metric

$$d((s,t),(s',t')) = |t-t'| + \sum_{j=1}^{d} |F_j(s_j) - F_j(s'_j)| \,,$$

and such that

Upper bound (to continue)

- $\mathbb{E}\Big(\sup_{\substack{s\in\mathbb{R}^d\\ k\le n}} |R_X(s,k) - K_X(s,k)|\Big) = O(n^{1/p}(\log n)^{\lambda(d)})\,,$

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- For any construction of a Kiefer process G_Y with covariance function Γ_Y

$$\sup_{1\le k\le 2^{N+1}} \sup_{s\in[0,1]^d} \left|R_Y(s,k) - G_Y(s,k)\right|$$
$$\le \sup_{s\in[0,1]^d} \left|R_Y(s,1) - G_Y(s,1)\right| + \sum_{L=0}^{N} D_L(G_Y)\,.$$

where

$$D_L(G_Y) := \sup_{2^L<\ell\le 2^{L+1}} \sup_{s\in[0,1]^d} \left|(R_Y(s,\ell)-R_Y(s,2^L))-(G_Y(s,\ell)-G_Y(s,2^L))\right|.$$

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- Reduction to a grid. Let A_n denote the set of x in $[0,1]^d$ such that nx is a multivariate integer. Let

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- We consider $d = 1$. Let $\vec{U}^{(0)}_{k,L} = \vec{U}_{k,L} - \mathbb{E}(\vec{U}_{k,L})$ where

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- Hence we have constructed a sequence of centered Gaussian random variables $(\vec{T}_L)_{L\in\mathbb{N}}$ in $\mathbb{R}^{2^{(d+1)L}}$ such that $\mathbb{E}\big(\vec{T}_L\vec{T}_L^t\big) = C_L$, and that are mutually independent.

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- Hence, in the independent case, there exists a Kiefer process K_Y with covariance Γ_Y such that for $s_{L,j} = (j_1 2^{-L},\dots,j_d 2^{-L})$

$$K_Y(s_{L,j}, \ell + 2^L) - K_Y(s_{L,j}, 2^L) = \big(\vec{T}_L\big)_{(\ell-1)2^{dL}+\sum_{i=1}^d (j_i-1)2^{(d-i)L}+1}\,.$$

(Dudley and Philipp (1983))

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- In the dependent case, much work is needed.

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$$\mathbb{E}\big(c_{(d+1)L}(\vec{S}_L, \vec{T}_L)\big) = \sup_{f \in \mathrm{Lip}(c_{(d+1)L})} \big(\mathbb{E}\big(f(\vec{S}_L)\big) - \mathbb{E}(f(\vec{T}_L))\big) \ll 2^{L/3} L^{\lambda(d)}$$

- In the dependent case, much work is needed.
- To bound suitably:

$$\mathbb{E} \sup_{f \in \mathrm{Lip}(c_{(d+1)L})} \big(\mathbb{E}\big(f(\vec{S}_L)|\mathcal{F}_{2^L}\big) - \mathbb{E}(f(\vec{T}_L))\big)$$

we use the Lindeberg method for the "conditional Wasserstein distance" and we introduce sparse vectors (whose "real" dimension is L^{d+1} and not $2^{L(d+1)}$!).

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and

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$$\beta_{2,Y}(k) = \sup_{i\ge j\ge k} \mathbb{E}(b(\mathcal{F}_0, i, j))\,.$$

- For the stationary Markov chain associated to the intermittent map considered before,

$$\beta_{2,Y}(k) = O(n^{1-p}) \text{ for any } p < 1/\gamma$$

(see Dedecker and Prieur (2009)).

Theorem : Dedecker, M., Rio (2012)

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$$\Lambda_X(s, s') = \sum_{k \geq 0} \mathrm{Cov}(\mathbf{1}_{X_0 \leq s}, \mathbf{1}_{X_k \leq s'}) + \sum_{k > 0} \mathrm{Cov}(\mathbf{1}_{X_0 \leq s'}, \mathbf{1}_{X_k \leq s})$$

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- Let $\Gamma_X(s, s', t, t') = \min(t, t')\Lambda_X(s, s')$. There exists a centered Gaussian process K_X with covariance function Γ_X, whose sample paths are almost surely uniformly continuous with respect to the pseudo metric

$$d((s,t),(s',t')) = |F(s) - F(s')| + |t - t'| \, ,$$

and such that for $\varepsilon = (p-2)^2/(22p^2)$,

$$\sup_{s \in \mathbb{R}, t \in [0,1]} |R_X(s, [nt]) - K_X(s, [nt])| = O(n^{1/2-\varepsilon}) \quad \text{almost surely,}$$

On the optimality of the result

- There exists a Markov chain such that $\beta_{2,X}(k) > ck^{-1}$ for some positive constant c such that the finite dimensional marginals of the process $\{(n \ln n)^{-1/2} R_T(\cdot, n)\}$ converge in distribution to those of the degenerated Gaussian process G defined by

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$$\text{for any } t \in [0,1], \quad G(t) = f(t)\mathbf{1}_{t \neq 0} Z\,,$$

  where Z is a standard normal.
- This shows that an approximation by a Kiefer process as in our main result cannot hold for this chain.

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